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DEFORMATION BOUNDS FOR A CREEPING STRUCTURE APPROACHING RUPTURE

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Abstract—The paper discusses the application of an energy principle [10] to constitutive relationships which describes the creep rupture of metals. It is shown that a bound on the displacement of a body subject to constant load, may be evaluated prior to the instant when rupture occurs at the most highly strained region. For a certain class of compact structures the bound is reduced to a simple form involving the rupture behaviour at a mean stress. An example of a two bar structure shows that the displacement bound may, in some circumstances, be translated into a lower bound on the time to initial rupture.

1. INTRODUCTION

THE lifetime of many engineering structures which operate at high temperatures depends upon the rupture property of the material. Continued creep deformation, which occurs in metals and alloys at temperatures above approximately half the melting temperature, produces degeneration of the material structure. Cracks and voids appear and ultimately failure occurs. The mode of failure may be a crack propagating through embrittled material or large plastic distortion of a plastically weakened material. Although the broad features of this behaviour have been known for some time, attempts to predict the behaviour of a body approaching rupture have occurred only in recent years. The problem requires, at the outset, a description of material behaviour which takes into account the degenerative effects of creep deformation. A most promising theory has been developed by Kachanov [1, 2], which is discussed in the recently translated book by Rabotnov [3] and the monograph by Odqvist [4]. In this theory the actual stress σ in a uniaxial specimen is assumed to be intensified by a damage function ω which represents the loss in effective cross-sectional area due to the presence of cracks and voids. Thus the conventional stationary state creep relationship

$$\dot{\varepsilon} = k\sigma^n \tag{1}$$

becomes

$$\dot{\varepsilon} = k\left(\frac{\sigma}{1-\omega}\right)^n, \tag{2}$$

where $\dot{\varepsilon}$ denotes the creep strain rate, k a material constant, and n the creep index. We will assume that n is an odd integer. When ω increases from zero the strain rate predicted by (2) exceeds that of (1) and becomes infinite when $\omega = 1$. The change in ω as time increases is predicted by a further state equation

$$\dot{\omega} = A\left|\frac{\sigma}{1-\omega}\right|^{\nu}, \tag{3}$$

where A and ν are material constants. A full description of the predictions of equations (2) and (3) may be found in Rabotnov's book [3], including some extensions. Here it suffices to mention that the predictions of creep rupture time (when $\omega = 1$), and the strains at which rupture occurs may be adequately predicted by equations (2) and (3) but the form of the strain–time curve is less faithfully reproduced.

Equations (2) and (3) may be generalized to a general state of stress in a variety of manners and the forms most appropriate to metals remains an open question at this time. A particularly simple form appears when it is assumed that the degeneration may be described by a scalar function and that the material remains isotropic (or retains a constant degree of anisotropy) as creep deformation proceeds. Such a model would be consistent with random void growth and cracking with no preferred directions developing in the material. In terms of a homogeneous function of the components of the stress tensor σ_{ij} of degree one, $\phi(\sigma_{ij})$, the generalization of (2) and (3) becomes

$$\dot{\varepsilon}_{ij} = k\frac{\partial}{\partial S_{ij}}\frac{\phi^{n+1}(S_{kl})}{n+1}, \tag{4}$$

and

$$\dot{\omega} = A|\phi(S_{ij})|^{\nu}, \tag{5}$$

where

$$S_{ij} = \sigma_{ij}/(1-\omega).$$

It appears unlikely that (4) and (5) have wide applicability, as the requirement that (4) shall coincide with the usual steady state creep relationship when $\omega = 0$ requires that ϕ shall depend only upon the deviatoric stress components and be independent of the mean hydrostatic stress. The consequence of the assumption that $\dot{\omega}$ is independent of mean hydrostatic stress, seems implausible in general. For certain metals, however, creep rupture appears to be governed by a maximum shear stress criterion (see for example Henderson [6] and Hayhurst [5]) indicating that equations (4) and (5) may be appropriate in some circumstances. The equations may well be of use in problems where the principle stress directions remain reasonably constant in time and where one stress component has the dominant effect. Equations (4) and (5) are introduced here for purely pragmatic reasons, which will become apparent late in the text.

The predictions of the behaviour of structures using these equations has occurred only fairly recently. Some simple examples are included in Refs. [3] and [4]. Recently Hayhurst [5] has investigated the behaviour of a plate containing a circular hole under tension, both experimentally and theoretically, using equations of a similar form. Solutions for parallel bar structure and a thick walled tube under internal pressure are included in the report of Martin and Leckie [7].

The problem of predicting the creep rupture of a structure has features which are not untypical of many structural calculations. To arrive at the quantity of interest, in this case the time when the full-scale rupture of the body occurs, necessitates the calculation of the complete history of stress and strain over the entire history of loading. Such calculations are often tedious and expensive in terms of effort. Martin and Leckie have investigated the possibility of predicting the rupture time without carrying out a complete analysis. In [7] they have described a method of calculating a lower bound to the rupture time for a class of parallel bar structures which involve the calculation of "modal" solutions to the problem. These "modal" solutions involve constant stress histories and are much simpler to evaluate than the complete solution. A further upper bound is described in [8].

The author has been involved in the development of general methods of approximate analysis of structures using energy methods. These methods allow the prediction of bounds on the displacement and deformation of structures, composed of inelastic materials, in terms of equilibrium stress fields. The theory is developed in its various aspects [9–13] for time dependent materials. The purpose of this paper is to recount an investigation into the application of these theorems to a body composed of the material described by equations (2), (3), (4) and (5). To achieve this end it becomes necessary to investigate the problems associated with unstable materials which fail to satisfy certain conditions which are required for the derivation of the bounds. It is shown that only limited information may be achieved and the prediction of the time to creep rupture due to plastic collapse or unbounded creep displacement remains beyond the scope of the theory. However, it is possible to derive an upper bound on the displacement of the structure from the time of initial loading to the time when rupture first occurs at some more highly strained point in the structure. This more limited objective is pursued analytically for the case $n = \nu$ to produce a direct relationship between the deformation of the structure and the plastic limit load associated with the yield condition

$$\phi(\sigma_{kl}) = \sigma_y,$$

where σ_y denotes the uniaxial yield stress. It is assumed that plastic behaviour of the structure involves the complete volume of the body being at yield at the limit state, and the theory is therefore confined to compact structures. It is then shown that for structures with a single kinematic redundancy this bound may be interpreted as a lower bound to the time when rupture first occurs at the most highly strained part of the structure. The bound is computed for a structure consisting of two parallel bars. Comparison with the analytic solutions indicate that the predictions are acceptably accurate.

In Section 2 the bounding method is described and the difficulties associated with unstable materials are discussed. In Section 3 the extremal stress histories involved in the bounding method are derived from the Katchanov equations. The analysis is first carried through for a uniaxial stress state and subsequently generalized in Section 4. In Section 5 the bounding method is derived in terms of the plastic limit state and in Section 6 the result is applied to a two bar structure.

2. THE APPLICATION OF AN ENERGY THEOREM TO UNSTABLE MATERIALS

The energy theorem [9, 10] concerns the behaviour of a body with volume V which is subject to time constant surface tractions P_i over part of the surface which we denote by S_T and suffers no displacement u_i over the remainder of the surface S_u.

The loads are applied at time $t = 0$ and the subsequent quasistatic deformation results in displacements $u_i(x_j, t)$. In [9] an upper bound on the total deformation was derived which, on ignoring elastic strains, may be expressed in the form

$$\int_{S_T} P_i u_i(T)\,dS \le \frac{1}{(\lambda-1)}\int_V \bar{w}(\lambda\sigma_{ij}^{g}, T)\,dV. \tag{6}$$

Here σ_{ij}^{g} denotes an arbitrary statically admissible stress field, in equilibrium with P_i on S_T, and λ is a parameter which may have any value $\lambda > 1$. The function $\bar{w}(\sigma_{ij}^*, T)$ provides the maximum value of the integral $\int_0^T \varepsilon_{ij}\dot{\sigma}_{ij}\,\mathrm{d}t$ amongst all possible stress histories which satisfy the conditions $\sigma_{ij}(0) = 0$ and $\sigma_{ij}(T) = \sigma_{ij}^*$. Thus

$$\bar{w}(\sigma_{ij}^*, T) \geq \int_0^T \varepsilon_{ij}\dot{\sigma}_{ij}\,\mathrm{d}t. \tag{7}$$

The derivation of (6) presupposes that the constitutive relationship satisfied three conditions.

(1) For a small instantaneous change in stress $\mathrm{d}\sigma_{ij}$, the resulting small instantaneous change in strain $\mathrm{d}\varepsilon_{ij}$ satisfies the Drucker stability condition [14]

$$\mathrm{d}\sigma_{ij}\,\mathrm{d}\varepsilon_{ij} \geq 0.$$

(2) At an arbitrary state of stress $\sigma_{ij}(T)$ which is the terminal stress of some stress history commencing at $\sigma_{ij}(0)$ it must be possible to instantaneously change the stress to the terminal state of any other similarly defined stress history.

(3) The maximum complementary energy function $\bar{w}$ must be finite. The condition (3) arises from purely practical considerations whereas (1) and (2) are sufficient conditions for the derivation of (5). These conditions are generally satisfied by constitutive relationships which describe the creep and plastic deformation of metals. The creep rupture relationships of the form (4) and (5) are however unstable in some sense and violate at least one of these conditions once rupture is allowed to occur.

To simplify arguments we first refer to the simple creep rupture model whose behaviour is described for uniaxial stress $\sigma > 0$, by

$$\dot{\varepsilon} = k\sigma^n, \qquad \varepsilon < \varepsilon_f;$$

and either,

$$\varepsilon \geq \varepsilon_f, \qquad \sigma = 0.$$

or

$$\dot{\varepsilon} \to \infty, \qquad \sigma > 0. \tag{8}$$

Rupture occurs at a critical strain ε_f prior to which the usual homogeneous relationship holds. When rupture occurs either the stress is zero or the creep rate is indeterminate.

The model contains the principal features of the Kachanov equations in a much simplified form. We now proceed to show that the conditions for (6) are contravened. This is most easily achieved by showing that the complementary work $\int_0^T \varepsilon\dot{\sigma}\,\mathrm{d}t$ may not be bounded from above. Consider the stress path shown in Fig. 1. Within the time interval $0 < t < t_1$ a sufficiently high constant stress is maintained to cause rupture at $t = t_1$. The stress is then reduced to zero and remains zero until a time $t_2 < T$. A finite strain $\bar{\varepsilon}$ is then produced by imposing a stress for an instant. The magnitude of $\bar{\varepsilon}$ may be as large as desired. The stress remains zero again until $t = T$ when the stress is instantaneously brought to its final value $\sigma^*(T)$. It is not necessary to calculate the complementary work in detail; it is sufficient to notice that the complementary work during $0 < t < t_1$ is finite and the contribution at $t = t_2$ may be made as small as we wish, as an infinitesimal small stress is sufficient to produce the strain $\bar{\varepsilon}$. The contribution from the stress change at $t = T$

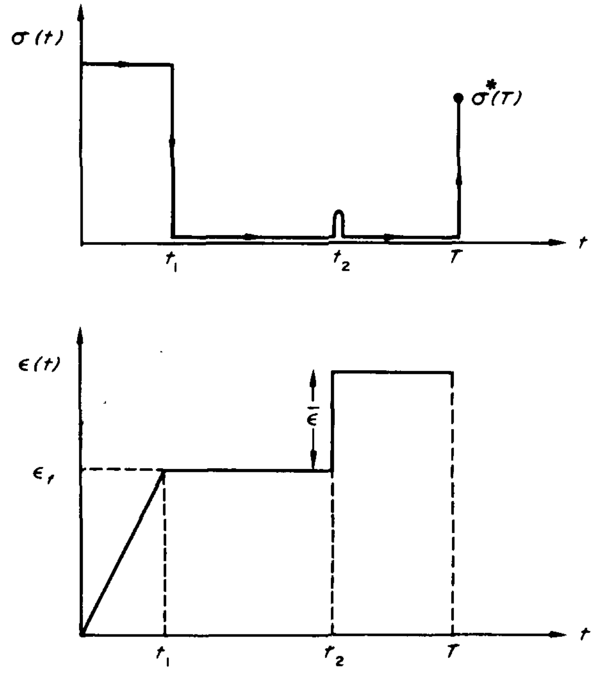

FIG. 1.

is given by $\bar{\varepsilon}\sigma^*(T)$. As $\bar{\varepsilon}$ may be made as large as we like, the complementary work may not be bounded from above. It may easily be shown that the Kachanov equations show the same mode of behaviour in this respect as equation (8), as the unbounded nature of the complementary work arises from the existence of a ruptured condition.

This difficulty may possibly be overcome by introducing a model in which final failure occurs by plastic collapse at a reduced yield stress. Such a model is provided by the equations

$$\dot{\varepsilon} = \dot{v} + \dot{p}$$

$$\left.\begin{aligned} \dot{v} = k\sigma^n, \dot{p} = 0, \sigma < \sigma_y f(v) \\ \dot{p} \geq 0, \sigma = \sigma_y f(v) \end{aligned}\right\}, \tag{9}$$

where $f(v)$ denotes a decreasing function of the creep strain v, and p denotes the plastic strain. A specimen of such a material will collapse plastically when $\sigma_y f(v)$ reduces to the applied stress. It may be shown that an increase in plastic strain at a fixed yield stress has the effect of decreasing the complementary work [10] and the effect shown in the previous model does not occur. Unfortunately this model contravenes condition (2). Consider two constant stress histories $\sigma_1 > \sigma_2$ which results in creep strain $v_1(t)$ and $v_2(t)$. The current yield values will differ, so that at some time t_1 there exists some stress σ_3 which lies between σ_1 and σ_2, which cannot be reached by the specimen which has been maintained at the higher stress level.

As a consequence of these calculations we see that the energy theorem (6) will not hold for the Kachanov equations, or a plastically softening material. We may, however, derive a theorem which requires less strict conditions on material behaviour. In [9] it was shown that the stress histories which maximize the complementary work between prescribed states of stress provides strain histories which minimize the work between prescribed states of strain. We may derive a displacement bound by commencing with the assumption that the work done between zero strain at $t = 0$ and strain $\varepsilon_{ij}(T)$ may be bounded from below. We postulate that it is possible to find a function of ε_{ij}, $w(\varepsilon_{ij}(T),T)$ so that

$$\int_0^T \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}t \geq w(\varepsilon_{kl}(T), T). \tag{10}$$

We now seek a function $\Omega(\varepsilon_{kl})$ which satisfies

$$w(\varepsilon_{kl}(T)) \geq \Omega(\varepsilon_{kl}(T)), \tag{11}$$

and which also satisfies the convexity condition

$$\Omega(\varepsilon^1_{ij}(T)) - \Omega(\varepsilon^2_{ij}(T)) - \left(\frac{\partial\Omega}{\partial\varepsilon_{ij}(T)}\right)_{\varepsilon^2_{ij}} \{\varepsilon^1_{ij}(T) - \varepsilon^2_{ij}(T)\} \geq 0. \tag{12}$$

Inequality (12) is equivalent to the condition that the stress–strain relationship

$$\sigma_{ij} = \frac{\partial\Omega}{\partial\varepsilon_{ij}} \tag{13}$$

shall have positive slope when expressed in terms of uniaxial stress and strain.

A complementary work function $\bar{\Omega}(\sigma_{ij})$ may now be defined by the relationship

$$\varepsilon_{ij} = \frac{\partial\bar{\Omega}}{\partial\sigma_{ij}}, \tag{14}$$

where (14) is the inverse of (13). The functions Ω and $\bar{\Omega}$ are related by

$$\Omega(\varepsilon_{kl}) + \bar{\Omega}(\sigma_{kl}) = \sigma_{ij}\varepsilon_{ij}. \tag{15}$$

The energy theorem may now be derived from inequalities (10), (11) and (12), which combine with equation (15) to yield the inequality

$$\int_0^{\varepsilon^1_{ij}(T)} \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}t \geq \sigma^2_{ij}\varepsilon^1_{ij} - \bar{\Omega}(\sigma^2_{kl}). \tag{16}$$

The strain ε^1_{ij} may be interpreted as the strain within the body at time $t = T$ and the integral on the left hand side may be taken over the history of strain which occurs at a point in the body during $0 \leq t \leq T$. We equate the stress σ^2_{ij} to $\lambda\sigma^g_{ij}$ where σ^g_{ij} denotes an arbitrary equilibrium stress field in equilibrium with P_i on S_T. Integrating inequality (16) over the volume V and applying the principle of virtual work, we obtain

$$\int_V \int_0^T \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}t\,\mathrm{d}V = \int_{S_T}\int_0^T P_i\dot{u}_i\,\mathrm{d}t\,\mathrm{d}S = \int_{S_T} P_i u(T)\,\mathrm{d}S$$

$$\geq \lambda \int_{S_T} P_i u_i(T)\,\mathrm{d}S - \int_V \bar{\Omega}(\lambda\sigma^g_{kl})\,\mathrm{d}V,$$

which, on rearrangement becomes

$$\int_{S_T} P_i u_i(T)\, dS \leq \frac{1}{\lambda - 1} \int_V \bar{\Omega}(\lambda \sigma_{kl}^{s})\, dV. \tag{17}$$

The inequalities (17) and (6) are identical except that the function $\bar{\Omega}$ has replaced $\bar{w}$. Clearly when $\bar{w}$, which results from minimum work histories, is also convex then the two inequalities are identical. The inequality (11) however allows a wider range of material behaviour, as the bound exists provided any finite Ω may be found which satisfy the inequalities (11) and (12).

We may now compute the minimum work histories for the models described by equations (8) and (9). The analysis will not be given in detail. Figure 2 shows the minimum work for uniaxial strain, together with the stress–strain relationship

$$\sigma = \frac{dw}{d\varepsilon},$$

for two times T_1 and T_2. It can be clearly seen that w is not convex and that there exists no non-zero convex function Ω. The model described by equation (9) exhibits the same behaviour, except that ε_f is replaced by the creep strain corresponding to $f(\varepsilon_f) = 0$ and the minimum work to states of strain $\varepsilon < \varepsilon_f$ is somewhat increased.

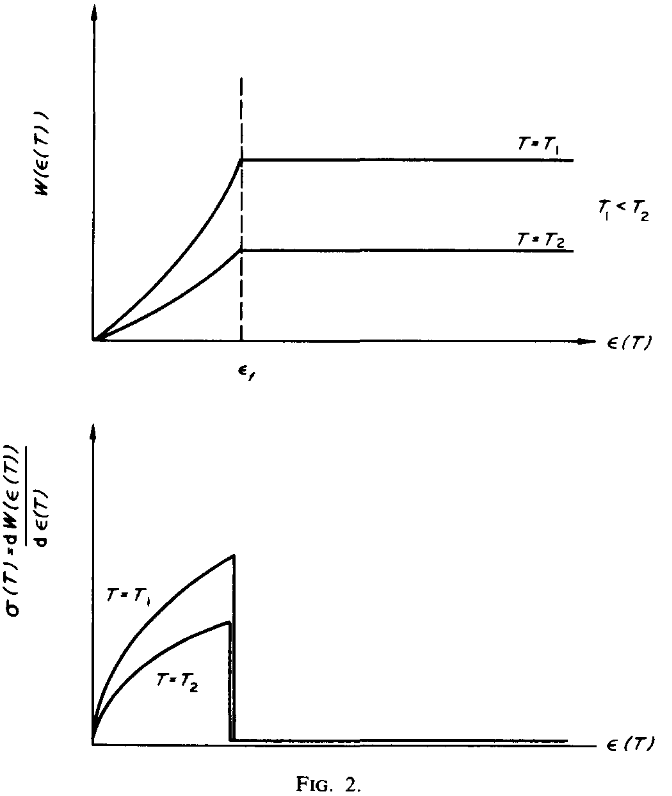

FIG. 2.

It may be concluded that it is unlikely that a theorem of the form of equation (17) exists for materials models which describe the creep rupture of metals once creep rupture has occurred at any point in the body. We may, however, develop a theorem if we assume that a rupture state has nowhere been achieved. For the models described by equations (8) and (9) such a theorem would be very uninformative, as it would merely produce the complementary energy theorem for equation (1). If, however, such a theorem is developed for the Kachanov equations (2) and (3), the effects of material degeneration will appear in the theorem. The theorem would remain valid while $\omega < 1$ throughout the volume and will cease to provide a bound once rupture commences.

In the next section the extremal paths for the Kachanov equations (2) and (3) are evaluated for a uniaxial state of stress, which are then generalized to the multiaxial case in Section 4.

3. THE EXTREMAL PATHS FOR THE KACHANOV EQUATIONS: THE UNIAXIAL CASE

As the conditions described in Section 2 are fulfilled by equations (2) and (3) provided $\omega < 1$, we may adopt the procedure described in [10]. The extremal paths are evaluated from the mimimum work condition, and we look for the history of strain $\varepsilon(t)$ which minimizes

$$F = \int_0^T \sigma\dot{\varepsilon}\,\mathrm{d}t - \mu\left\{\int_0^T \dot{\varepsilon}\,\mathrm{d}t - \varepsilon(T)\right\}. \tag{18}$$

where μ denotes a Lagrange multiplier. This objective may be achieved by re-expressing equation (18) in terms of the damage function ω, to yield

$$\begin{aligned} F &= -\int_0^T \psi(\omega, \dot{\omega})\,\mathrm{d}t \\ \psi(\omega, \dot{\omega}) &= kA^{-n/\nu}\{\mu\dot{\omega}^{n/\nu} - A^{-1/\nu}\dot{\omega}^{(n+1)/\nu}(1-\omega)\}, \qquad \dot{\omega} > 0. \end{aligned} \tag{19}$$

We seek the minimum of equation (19) amongst the damage histories $\omega(t)$ which satisfy $\omega(0) = 0$. The first variation of F is given by

$$\begin{aligned} \delta F &= -\int_0^T \left(\frac{\partial\psi}{\partial\omega}\delta\omega + \frac{\partial\psi}{\partial\dot{\omega}}\delta\dot{\omega}\right)\mathrm{d}t \\ &= -\left[\frac{\partial\psi}{\partial\dot{\omega}}\delta\omega\right]_0^T + \int_0^T \left\{\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial\psi}{\partial\dot{\omega}}\right\} - \frac{\partial\psi}{\partial\omega}\right\}\delta\omega\,\mathrm{d}t. \end{aligned} \tag{20}$$

The condition $\delta F = 0$ gives rise to the natural boundary condition, on noting that $\delta\omega(0) = 0$,

$$\left[\frac{\partial\psi}{\partial\dot{\omega}}\right]_{t=T} = 0, \tag{22}$$

and the Euler equation

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\frac{\partial\psi}{\partial\dot{\omega}}\right\} - \frac{\partial\psi}{\partial\omega} = 0. \tag{23}$$

Equation (22) yields that

$$\mu = \frac{n+1}{n} A^{-1/\nu} \dot{\omega}(T)^{1/\nu} (1-\omega(T)) = \frac{n+1}{n} \sigma(T). \tag{24}$$

The Euler equation (23) may be integrated on noting that

$$\frac{\mathrm{d}}{\mathrm{d}t}\left\{\dot{\omega}\frac{\partial\psi}{\partial\dot{\omega}} - \psi\right\} = \dot{\omega}\left\{\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial\psi}{\partial\dot{\omega}}\right) - \frac{\partial\psi}{\partial\omega}\right\},$$

so that

$$\dot{\omega}\frac{\partial\psi}{\partial\dot{\omega}} - \psi = \text{constant}. \tag{25}$$

Substituting for ψ in (25) yields

$$\frac{n-\nu}{\nu}\mu\dot{\varepsilon}(t) - \left(\frac{n+1-\nu}{\nu}\right)\sigma(t)\dot{\varepsilon}(t) = \text{constant} \tag{26}$$

where μ is given by (24).

In its general form equation (26) does not appear to possess a simple analytic solution. However, when $n = \nu$ a simple solution does exist and we will pursue the analysis for this case. Note that this assumption allows equations (2) and (3) to be integrated to give $\omega(t)$ as a function of $\varepsilon(t)$,

$$\varepsilon(t) = \frac{k}{A}\omega(t), \tag{27}$$

and that the strain at rupture $\varepsilon = k/a$ remains independent of both stress history and the value of n. When $n = \nu$, equation (26) becomes

$$\sigma\dot{\varepsilon} = \text{constant}. \tag{28}$$

On substituting for $\sigma(t)$ and $\dot{\varepsilon}(t)$, (28) becomes

$$\dot{\omega}^{(n+1)/n}(1-\omega) = \text{constant},$$

which possesses the solution

$$(1-\omega) = (1-Ct)^{(n+1)/(2n+1)}, \tag{29}$$

where C denotes a constant of integration. The stress and strain histories resulting from equation (29) are given by

$$\begin{aligned} \sigma(t) &= \left[\frac{C}{A}\left(\frac{n+1}{2n+1}\right)\right]^{1/n}(1-Ct)^{n/(2n+1)}, \\ \dot{\varepsilon}(t) &= k\frac{C}{A}\left(\frac{n+1}{2n+1}\right)(1-Ct)^{-n/(2n+1)}. \end{aligned} \tag{30}$$

The stress history may be much simplified on substituting

$$C = \frac{1}{\bar{T}} = A\sigma^n(0)\left(\frac{2n+1}{n+1}\right), \tag{31}$$

to yield

$$\sigma(t) = \sigma(0)\left(1-\frac{t}{\bar{T}}\right)^{n/(2n+1)}. \tag{32}$$

Before the maximum complementary work may be computed we may note that at $t = T$ a discontinuity in the stress history is likely to occur [9, 10]. The value of this discontinuity may be evaluated from the minimum work by the introduction of an elastic strain [10]

$$\varepsilon(t) = \frac{\sigma(t)}{E} + v(t) = e(t) + v(t)$$

where $e(t)$ denotes the elastic strain and $v(t)$ denotes the inelastic strain. We are required to find the minimum of

$$F = \tfrac{1}{2}Ee^2 + \int_0^T \sigma\dot{v}\,\mathrm{d}t + \mu(e(T)+v(T)-\varepsilon(T))$$

where μ denotes a Lagrange multiplier. The minimum condition

$$\frac{\partial F}{\partial e(T)} = \frac{\partial F}{\partial v(T)} = 0$$

results in the relationship

$$\sigma(T) = \frac{\mathrm{d}w(v(T))}{\mathrm{d}v(T)}, \tag{33}$$

where $w(v(T))$ denotes the minimum work to the creep strain $v(T)$. From equation (30) we find

$$w = \sigma\dot{v}T = k\left(\frac{C}{A}\left(\frac{n+1}{2n+1}\right)\right)^{(n+1)/n} \tag{34}$$

on noting that

$$\frac{\mathrm{d}w}{\mathrm{d}v(T)} = \frac{\mathrm{d}w}{\mathrm{d}C}\,\frac{\mathrm{d}C}{\mathrm{d}\omega(T)}\,\frac{\mathrm{d}\omega(T)}{\mathrm{d}v(T)}$$

we obtain from equations (29), (30), (32), (33) and (34) after some algebra

$$\sigma(T) = \frac{n+1}{n}\sigma(T^-)$$

where $\sigma(T^-)$ denotes the terminal point of the continuous part of the stress history:

$$\sigma(T^-) = \sigma(0)\left(1-\frac{T}{\bar{T}}\right)^{n/(2n+1)}. \tag{35}$$

We may now evaluate the maximum complementary energy, which is found, after some algebra to be given by

$$\bar{w}(\sigma(T)) = \frac{k}{A}\,\frac{n+1}{n}[\sigma(T^-)-\sigma(0)+A\sigma^{n+1}(0)T]. \tag{36}$$

Unfortunately $\bar{w}(\sigma(T))$ is only an implicit function of $\sigma(T)$ as the relationship between $\sigma(T)$ and $\sigma(0)$, which is obtained from equations (30) and (36), may not be inverted analytically.

Finally, to show that the solution (26) of the Euler equation (23) provides a minimum it is necessary to show that the second variation of F is always positive. The analysis is given in Appendix 1 where it is shown that $\delta^2 F$ is always positive.

An expansion of equation (36) as a power series in A demonstrates the effect of the damage upon $\bar{w}(\sigma(T))$,

$$\bar{w}(\sigma(T)) = \frac{1}{n}k\sigma^{n+1}(0)T - kA\sigma^{2n+1}(0)T^2 - \tfrac{1}{2}A^2\sigma(0)^{3n+1}\frac{3n+1}{3(n+1)}T^3 + 0(A^3).$$

In the limit as $A \to 0$,

$$\bar{w}(\sigma(T)) = \frac{1}{n}k\sigma^{n+1}(0)T$$

and

$$\sigma(0) = \sigma(T^-) = \frac{n}{n+1}\sigma(T)$$

and we recover the result for equation (1) given in [9].

4. THE EXTREMAL PATHS FOR THE KACHANOV EQUATIONS: THE MULTIAXIAL CASE

In this section the theory of the previous section is rederived for the general creep relationships

$$\dot{\varepsilon}_{ij} = k\frac{\partial}{\partial S_{ij}}\left\{\frac{\phi^{n+1}(S_{ij})}{n+1}\right\} \tag{4}$$

$$\dot{\omega} = A|\phi(S_{ij})|^{\nu} \tag{5}$$

where

$$S_{ij} = \sigma_{ij}/(1-\omega).$$

The analysis of the uniaxial case requires only slight extension as the equation for $\omega(t)$ will be shown to be unchanged. Following the argument of Section 3, the function F becomes

$$F = \int_0^T \sigma_{ij}\dot{\varepsilon}_{ij}\,\mathrm{d}t - \mu_{ij}\left\{\int_0^T \dot{\varepsilon}_{ij}\,\mathrm{d}t - \varepsilon_{ij}(T)\right\}, \tag{37}$$

where μ_{ij} denotes a tensor of Lagrange multipliers. As the function ϕ is homogeneous of degree one the rate of work done becomes, upon substituting for ϕ from (5)

$$\sigma_{ij}\dot{\varepsilon}_{ij} = S_{ij}\frac{\partial}{\partial S_{ij}}\left\{\frac{\phi^{n+1}}{n+1}\right\}(1-\omega) \tag{38}$$

$$= (1-\omega)\phi^{n+1} = (1-\omega)\left(\frac{\dot{\omega}}{A}\right)^{(n+1)/\nu}.$$

Further

$$\mu_{ij}\dot{\varepsilon}_{ij} = \left(\mu_{ij}\frac{\partial \phi}{\partial S_{ij}}\right)\left(\frac{\dot{\omega}}{A}\right)^{n/v}. \tag{39}$$

The functional F now becomes

$$F = -\int_0^T \psi(\omega, \dot{\omega}, S_{ij})\,\mathrm{d}t \tag{40}$$

where

$$\psi(\omega, \dot{\omega}, S_{ij}) = \left\{-(1-\omega)\left(\frac{\dot{\omega}}{A}\right)^{(n+1)/v} + \left(\mu_{ij}\frac{\partial \phi}{\partial S_{ij}}\right)\left(\frac{\dot{\omega}}{A}\right)^{n/v}\right\}. \tag{41}$$

Note that ψ is identical to the corresponding uniaxial functional, equation (19), except that μ has been replaced by $\mu_{ij}(\partial\phi/\partial S_{ij})$. The first variation of F is given by

$$\delta F = \int_0^T \left\{\frac{\partial \psi}{\partial \omega}\delta\omega + \frac{\partial \psi}{\partial \dot{\omega}}\delta\dot{\omega} + \frac{\partial \psi}{\partial S_{kl}}\delta S_{kl}\right\}\mathrm{d}t. \tag{42}$$

The last term in the integral of (42) becomes

$$\frac{\partial \psi}{\partial S_{kl}}\delta S_{kl} = \left(\frac{\dot{\omega}}{A}\right)^{n/v}\mu_{ij}\frac{\partial^2 \phi}{\partial S_{kl}\partial S_{ij}}\delta S_{kl}. \tag{43}$$

We now proceed to show that $\delta F = 0$ if

$$\sigma_{ij} = (1-\omega(t))S_{ij} = \mu(t)\sigma_{ij}(T^-) \tag{44}$$

and

$$\mu_{ij} = \frac{n+1}{n}\sigma_{ij}(T^-)$$

where $\omega(t)$ is given by (23). Consider the coefficient of δS_{kl},

$$\mu_{ij}\frac{\partial^2 \phi}{\partial S_{kl}\partial S_{ij}} = \frac{n+1}{n}\frac{(1-\omega(t))}{\mu(t)}S_{ij}\frac{\partial^2 \phi}{\partial S_{ij}\partial S_{kl}}.$$

As ϕ is homogeneous of degree one, the derivative $\partial\phi/\partial S_{kl}$ will be homogeneous of degree zero for each component S_{kl}. Therefore, by Euler's theorem for a homogeneous function

$$S_{ij}\frac{\partial}{\partial S_{ij}}\left\{\frac{\partial \phi}{\partial S_{kl}}\right\} \equiv 0$$

and the coefficient of δS_{kl} is zero. The remaining terms in the integral of (42) now become identical to the uniaxial case as

$$\mu_{ij}\frac{\partial \phi}{\partial S_{ij}} = \frac{n+1}{n}\sigma_{ij}(T)\frac{\partial \phi}{\partial \sigma_{ij}(T)} = \frac{n+1}{n}\phi(\sigma_{ij}(T)) \tag{45}$$

where $\phi(\sigma_{ij}(T)$ replaces $\sigma(T)$ in (24). Similarly from (44) we may write

$$\phi(\sigma_{ij}(t)) = \mu(t)\phi(\sigma_{ij}(T^-)).$$

The analysis of the multiaxial case becomes identical to the uniaxial case except that $\sigma(t)$ is replaced by $\phi(\sigma_{ij}(t))$. The solution of the Euler equation becomes

$$\frac{n-\nu}{\nu}\left(\frac{n+1}{n}\right)\sigma_{ij}(T)\dot{\varepsilon}_{ij}(t)-\left(\frac{n+1-\nu}{\nu}\right)\sigma_{ij}(t)\dot{\varepsilon}_{ij}(t) = \text{constant}.$$

For the case $n = \nu$ we obtain

$$(1-\omega(t)) = (1-C't)^{(n+1)/(2n+1)}$$

and

$$\phi(\sigma_{ij}(t)) = \phi(\sigma_{ij}(0))\left[1-\frac{t}{\bar{\bar{T}}}\right]^{n/(2n+1)} \tag{46}$$

where

$$\frac{1}{\bar{\bar{T}}} = A\phi^{n}(\sigma_{ij}(0))\frac{2n+1}{n+1}. \tag{47}$$

The maximum complementary energy is given by

$$\bar{w}(\sigma_{ij}(T)) = \frac{k}{A}\frac{n+1}{n}[\phi(\sigma_{ij}(T^{-}))-\phi(\sigma_{ij}(0))+A\phi^{n+1}(\sigma_{ij}(0))T] \tag{48}$$

$$\sigma_{ij}(T) = \frac{n+1}{n}\sigma_{ij}(T^{-}).$$

5. THE DEFORMATION BOUND IN TERMS OF THE LIMIT LOAD SOLUTION

In the previous section the deformation bound

$$\int_{S_T} P_i u_i(T)\,\mathrm{d}S \leq \frac{1}{\lambda-1}\int_V \bar{w}(\lambda\sigma_{ij}^{g})\,\mathrm{d}V \tag{6}$$

has been derived in explicit form by the evaluation of the maximum complementary work $\bar{w}$ for the Kachanov equation. We repeat here the value of $\bar{w}$,

$$\bar{w}(\sigma_{ij}(T)) = \frac{k}{A}\frac{n+1}{n}[\phi(\sigma_{ij}(T^{-}))-\phi(\sigma_{ij}(0))+A\phi^{n+1}(\sigma_{ij}(0))T] \tag{48}$$

where

$$\phi(\sigma_{ij}(T^{-})) = \phi(\sigma_{ij}(0))\left[1-\frac{T}{\bar{\bar{T}}}\right]^{n/(2n+1)} \tag{46}$$

and

$$\sigma_{ij}(T) = \frac{n+1}{n}\sigma_{ij}(T^{-}).$$

Although the functions involved are of a fairly simple form, it is not possible to express $\bar{w}$ as an explicit function of its principal argument $\sigma_{ij}(T)$. The application of (6) requires the choice of a particular σ^g_{ij} and λ from which the related stresses $\sigma_{ij}(T^-)$ and $\sigma_{ij}(0)$ are computed for each point in the body. Certainly the computation is much simpler than a complete analysis of the problem, but optimization of the bound with respect to σ^g_{ij} and λ will be a time-consuming operation. In a previous paper [12] it was shown that the form of bounds of this type could be considerably simplified by equating σ^g_{ij} to the perfectly plastic limit state solution, giving a deformation bound which required, for some structures, knowledge only of the limit load and not the details of the limit state stress distribution. The results were principally relevant to "compact" structures, which, at the plastic limit state, are at a state of incipient plastic yield throughout the volume. Many of the simpler structures, such as beams under flexural, axial and torsional loading, thick tubes under internal pressure and plates under certain states of lateral and in plane loading, fall within this category. Less kinematically restrained structures, such as portal frames and near statically determinate pin-jointed frames, fall outside this category.

The adoption of a similar procedure for the theorem contained in equations (6), (46) and (48) above produces an extremely simple bound which we now derive. We note that although λ is a spatial constant, it may vary with time T. We are concerned with the interrelated stresses σ^g_{ij}, $\sigma_{ij}(T)$, $\sigma_{ij}(T^-)$ and $\sigma_{ij}(0)$ and their relationship between each other is shown schematically in Fig. (3). The function $\bar{w}$ (48) is an implicit function of $\sigma_{ij}(0)$ and we therefore require a simple means of relating $\sigma_{ij}(0)$ to σ^g_{ij}. We may achieve this by adjusting $\lambda(T)$ so that at all times

$$\sigma^g_{ij} = \sigma_{ij}(0)$$

i.e.

$$\sigma_{ij}(T^-) = \frac{n}{n+1}\lambda\sigma^g_{ij} = \frac{n}{n+1}\lambda\sigma_{ij}(0). \tag{49}$$

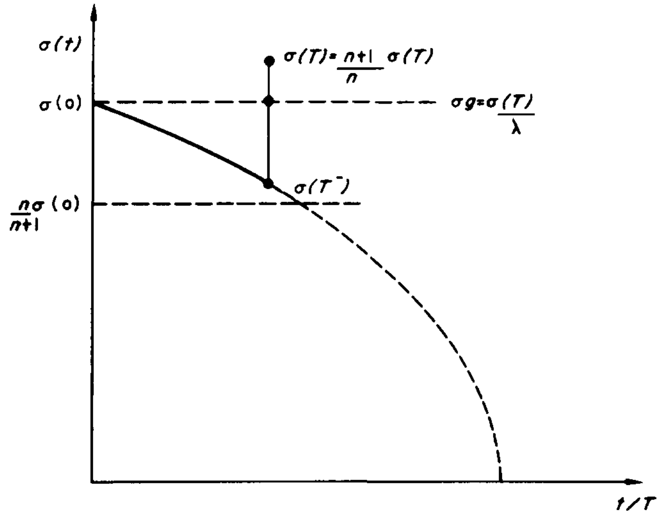

FIG. 3.

This procedure may continue until $\lambda(T) = 1$ when the bound (6) becomes infinite. The appropriate value of $\lambda(T)$ is given by substituting equation (49) into equation (46).

$$\lambda(T) = \frac{n+1}{n}\left[1-\frac{T}{\bar{\bar{T}}}\right]^{n/(2n+1)}, \quad \text{and} \quad \frac{1}{\bar{\bar{T}}} = A\phi^n(\sigma^g_{ij})\frac{2n+1}{n+1} \tag{50}$$

from which the range of T may be computed for a particular σ^g_{ij}.

For a general stress distribution σ^g_{ij} this computed value of $\lambda(T)$ would vary from point to point within the body. We may evaluate a constant value of $\lambda(T)$ if a stress distribution σ^g_{ij} exists for which $\phi(\sigma^g_{ij})$ remains constant throughout the volume. Such a distribution is provided by the limit state solution σ^L_{ij} associated with yield condition

$$\phi(\sigma_{ij}) - \sigma_y = 0 \tag{51}$$

where σ_y denotes the uniaxial yield stress.

Consider a class of loading states lP_i where l denotes a loading parameter. There will exist a particular value of $l = l_L$ at which plastic collapse will occur for a material satisfying the yield condition (51). The limit state stress distribution σ^L_{ij} will satisfy equation (51) for at least part of the volume, and we will assume that the yield condition is in fact satisfied throughout the volume and write

$$\sigma^g_{ij} = \frac{l}{l_L}\sigma^L_{ij} \tag{52}$$

so that

$$\phi(\sigma^g_{ij}) = \frac{l}{l_L}\phi(\sigma^L_{ij}) = \frac{l}{l_L}\sigma_y = \sigma_R \tag{53}$$

where σ_R denotes a mean stress for the structure. Thus we obtain from (50) the relationship

$$\frac{1}{\bar{\bar{T}}} = A\sigma^n_R\frac{2n+1}{n+1} \tag{54}$$

and $\lambda(T)$ becomes a function of σ_R and n. Further $\bar{w}$, equation (18), becomes a constant throughout the volume, and upon substituting the relationships (46) and (54) into (48) the bound (6) becomes

$$\int_{S_T} P_i u_i(T)\, \mathrm{d}S \leq \frac{k}{A}\sigma_R V f\left(n, \frac{T}{\bar{\bar{T}}}\right) \tag{55}$$

where V denotes the volume of the body and

$$f\left(n, \frac{T}{\bar{\bar{T}}}\right) = \frac{1}{\lambda(T)-1}\left[\lambda - \frac{n+1}{n} + \frac{T}{\bar{\bar{T}}}\frac{(n+1)^2}{n(2n+1)}\right] \tag{56}$$

where $\lambda(T)$ is given by equation (50).

The bound (55) requires knowledge of σ_R only, which may be computed from l_L. In the following section we investigate the application of the bound to a simple two bar structure.

6. THE APPLICATION OF THE BOUND TO STRUCTURES

We first compare the upper bound with the analytic behaviour of a bar subject to uniaxial constant stress σ. By integrating equations (2) and (3) it may be shown that the uniaxial strain for $n = v$ is given by

$$\varepsilon(t) = \frac{k}{A}\left[1-\left(1-\frac{(n+1)^2}{2n+1}\frac{T}{\overline{T}}\right)^{1/(n+1)}\right], \tag{56}$$

where

$$\frac{1}{\overline{T}} = A\sigma^n\left(\frac{2n+1}{n+1}\right);$$

whereas the upper bound yields, on equating $\sigma = \sigma_R$,

$$\varepsilon(t) \le \frac{k}{A}f\left(n, \frac{T}{\overline{T}}\right). \tag{57}$$

Equation (56) and bound (57) are shown for $n = 3$ and $n = 5$ in Figs. (4) and (5). It is interesting to note that the two curves are coincident at the rupture time $T/\overline{T} = (2n+1)/(n+1)^2$

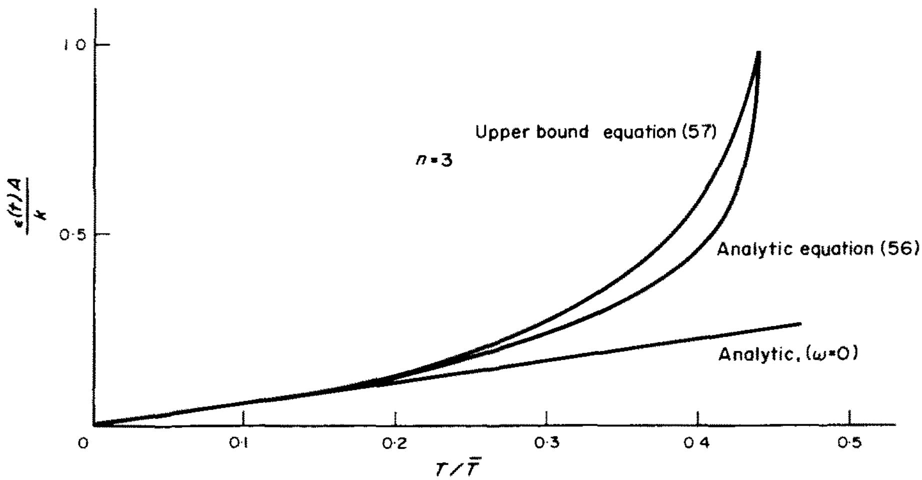

FIG. 4.

when $\varepsilon(t) = k/A$. The error before rupture is encouragingly small. It may be noted that a knowledge of the rupture strain and the upper bound allows a completely accurate calculation of the rupture time. Furthermore, we see that the bound (55) may be interpreted as

$$\int_{S_T} P_i u_i(T)\, \mathrm{d}S \le \sigma_R \bar{\varepsilon}(\sigma_R, T)V$$

where $\bar{\varepsilon}(\sigma_R, T) = (k/A)f(n, T/\overline{T})$ denotes the upper bounds on the uniaxial creep curves exhibited in Figs. (4) and (5).

We now proceed to compare the bound with the behaviour of the parallel bar structure shown in Fig. (6) which consists of two bars of equal cross-sectional area a and of length

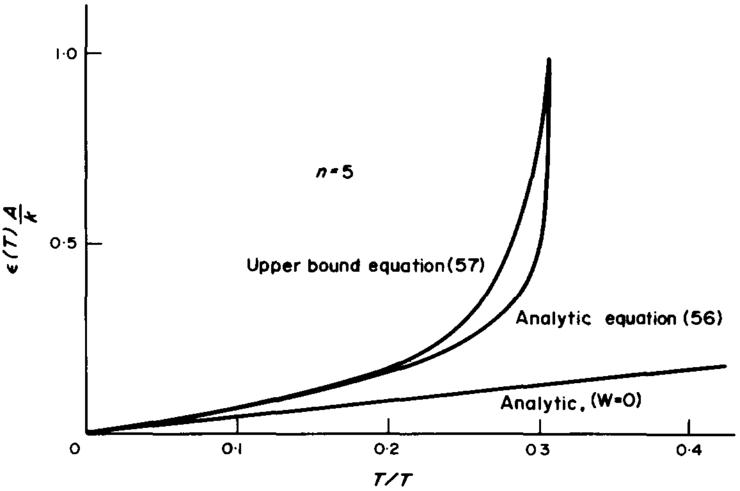

FIG. 5.

$L_1 = L_0$ and $L_2 = 8L_0$. This problem has been solved analytically for the equations (2) and (3) by Leckie and Martin [8]. The displacement of the model at time t for load P is given by

$$u(t) = \frac{P^{n-\nu}kL_0}{A(\nu+1)} \frac{\left(\sum \frac{1}{l_j}\right)^{\nu+1-n}}{\left(\sum \frac{1}{l_j^{\nu+1}}\right)} \{1-(1-A't)^{(\nu+1-n)/(\nu+1)}\},$$

$$A' = \frac{A(\nu+1)P^{\nu}}{\left(\sum \frac{1}{l_j}\right)^{\nu+1}} \left(\sum \frac{1}{l_j^{\nu+1}}\right), \tag{58}$$

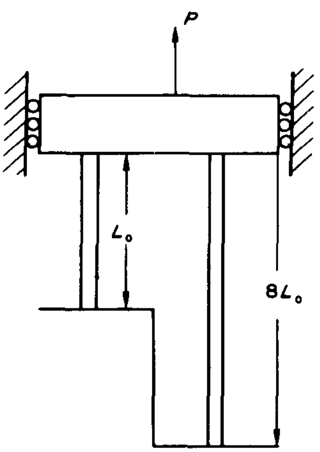

FIG. 6.

where

$$l_1 = \left(\frac{L_1}{L_0}\right)^{1/n} \quad \text{and} \quad l_2 = \left(\frac{L_2}{L_0}\right)^{1/n}.$$

As the limit load solution is given by

$$P_L = 2\sigma_y a$$

then

$$\sigma_R = \frac{P}{2a},$$

and the upper bound becomes

$$2\sigma_R a u(T) \leq \frac{k}{A}\sigma_R V f\left(n, \frac{T}{\bar{T}}\right), \frac{1}{\bar{T}} = A\sigma_R^n\left(\frac{2n+1}{n+1}\right).$$

Therefore

$$\frac{u(T)Aa}{Vk} \leq \tfrac{1}{2} f\left(n, \frac{T}{\bar{T}}\right). \tag{59}$$

The bound (59) and the analytic solution (58) are compared in Fig. 7 for $n = 3$ to the time when initial rupture occurs in the shorter bar. The predictions of equation (1) are also included, to indicate the additional displacement due to the presence of damage.

The most remarkable feature of Fig. 7 is that the analytic solution and the upper bound almost (but not exactly) coincide at the instant when initial rupture occurs in the smaller bar, which provides the limit of validity of the upper bound. As the strain at rupture $\varepsilon = k/A$ is known *a priori* for the case $n = \nu$, and the displacement $u = (k/A)L_0$ is also known from consideration of kinematics, it is possible to obtain a lower bound on the

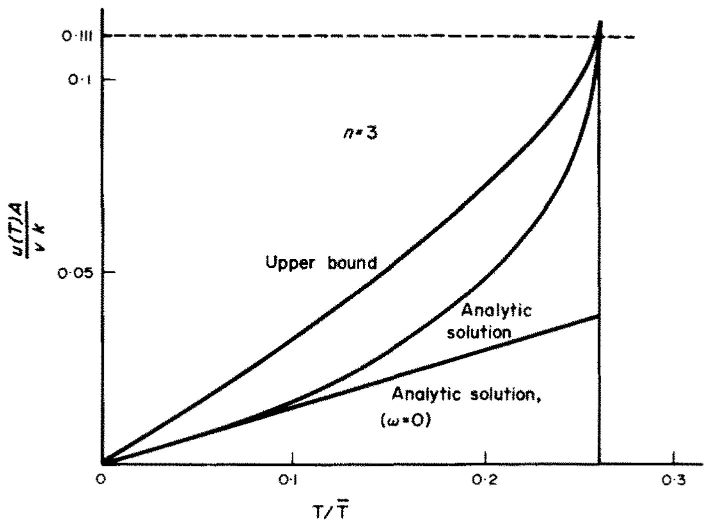

FIG. 7.

initial rupture time as the time when the upper bound equals this value. For the case shown in Fig. 7 this estimate is extremely close to the analytic value.

This procedure may clearly be applied to any structure where the maximum strain is determined by the surface displacements. Further discussion of the application of the bounds will appear in a forthcoming paper.

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APPENDIX 1

The complementary energy is given by

$$F = \int_0^T (\mu - \sigma(t))\dot{\varepsilon}(t)\,dt = \int_0^T \phi(\omega, \dot{\omega})\,dt$$

where

$$\phi = \{\mu k \dot{\omega}^{n/m} - \dot{\omega}^{(n+1)/m}(1-\omega)\} k A^{-n/\nu}. \tag{A1}$$

For the second variation of $\bar{w}$ will be negative provided the following inequalities hold

$$\frac{\partial^2 \phi}{\partial \dot{\omega}^2} < 0 \tag{A2}$$

and

$$\left(\frac{\partial^2 \phi}{\partial \dot{\omega}^2}\right)\left(\frac{\partial^2 \phi}{\partial \omega^2}\right) - \left(\frac{\partial^2 \phi}{\partial \omega\, \partial \dot{\omega}}\right)^2 < 0. \tag{A3}$$

Clearly (A3) is always satisfied for $\omega \neq 0$ as ϕ is a linear function of ω. For $n = \nu$ and $\mu = [(n+1)/n]\sigma(T)$.

$$\frac{\partial^2 \phi}{\partial \dot{\omega}^2} = -\frac{n+1}{n^2}\dot{\omega}^{-n/(n+1)}(1-\omega)kA^{-n/\nu}.$$

As $\dot{\omega} > 0$ and $\omega < 1$ we see that (A2) is always satisfied. Thus $\delta^2 \bar{w} < 0$ for arbitrary variations in ω and $\dot{\omega}$ and we conclude that $\bar{w}$ can possess neither a minimum or a saddle point.

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Абстракт—Работа обсуждает применение принципа энергии (10) для конститутивных соотношений, которое описывает разрушение металлов вследствие ползучести. Указано, что предел на перемещениях тела, полверженного действию постоянной нагрузки, можно определить предварительно к моменту, когда происходит разрушение в найболее напряженном районе. Для некоторого класса компактных конструкций, предел сводится к простой форме, заключая поведение разрушения для среднего напряжния. Пример конструкции, состоящей из двух стержней, указывает, что можно, при некоторых, условиях, перенсти параллельно границу перемещений к нижному пределу, во время начального разрушения.